

Exercise Sheet 3

Exercise 1. Let G be a Hausdorff topological group acting continuously on a Hausdorff space X . Let

$$\rho : G \times X \rightarrow X \times X, \quad \rho(g, x) = (gx, x).$$

For a compact $K \subset X$, put

$$G_K = \{g \in G \mid gK \cap K \neq \emptyset\}.$$

- (1) Show that $\rho(G \times X)$ is closed if and only if X/G is Hausdorff.
- (2) Show that ρ is proper if and only if G_K is compact for every compact $K \subset X$.
Hint: Take $L \subset X \times X$ compact and set $K_1 = \pi_1(L)$, $K_2 = \pi_2(L)$. Show that $\rho^{-1}(L) \subset G_{K_1 \cup K_2} \times K_2$.
- (3) Suppose G is discrete. Show that ρ is proper if and only if G_K is finite for every compact $K \subset X$.

Exercise 2. Give an example of a non-proper action of \mathbb{Z} on the complex plane by biholomorphisms.

Exercise 3. Let $\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$ and $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$ act on \mathbb{H} by

$$z \mapsto \gamma z = \frac{az + b}{cz + d} \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let

$$\mathcal{F} = \left\{ z \in \mathbb{H} : -\frac{1}{2} \leq \Re z \leq \frac{1}{2}, |z| \geq 1 \right\}.$$

- (1) Show that \mathbb{H} is preserved by the action of $\mathrm{PSL}_2(\mathbb{Z})$.
- (2) If $z_1, z_2 \in \mathcal{F}$ are distinct and $z_2 = \gamma z_1$ for some $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, prove that either $\Re z_1 = \pm \frac{1}{2}$ and $z_2 = z_1 \pm 1$, or $|z_1| = 1$ and $z_2 = -1/\bar{z}_1$.
- (3) Deduce that \mathcal{F} contains exactly one representative of each $\mathrm{PSL}_2(\mathbb{Z})$ -orbit (up to boundary points). Such a \mathcal{F} is called a fundamental domain.
- (4) Show that the action of $\mathrm{PSL}_2(\mathbb{Z})$ on \mathbb{H} is proper.

Exercise 4. (for credit, due on 5 October) Let $\omega, \omega' \in \mathbb{H}$ and $\Lambda = \mathbb{Z} + \omega\mathbb{Z}$, $\Lambda' = \mathbb{Z} + \omega'\mathbb{Z}$. Let $\pi : \mathbb{C} \rightarrow \mathbb{C}/\Lambda$ and $\pi' : \mathbb{C} \rightarrow \mathbb{C}/\Lambda'$ be the quotient maps.

- (0) (0.5 points) Let $\mu(\Lambda) = \inf\{|\lambda| : \lambda \in \Lambda \setminus \{0\}\}$. Note that $\mu(\Lambda) > 0$. For any $r < \mu(\Lambda)/2$ and any $z_0 \in \mathbb{C}$, show that the restriction

$$\pi|_{B_r(z_0)} : B_r(z_0) \rightarrow \mathbb{C}/\Lambda$$

is injective.

- (1) (0.5 points) Show that \mathbb{C}/Λ is homeomorphic to the torus $S^1 \times S^1$.
- (2) (0.5 points) Endow \mathbb{C}/Λ with a structure of a Riemann surface.
- (3) (2 points) Show that any holomorphic map $f : \mathbb{C}/\Lambda \rightarrow \mathbb{C}/\Lambda'$ lifts to an affine map $F(z) = az + b$ with $a\Lambda \subset \Lambda'$. **Hint:** Lift f to a holomorphic map F , whose derivative is Λ -periodic, and apply Liouville's theorem.
- (4) (1.5 points) Show that \mathbb{C}/Λ is biholomorphic to \mathbb{C}/Λ' if and only if there exist $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$ such that $\omega' = \frac{a\omega + b}{c\omega + d}$.

Exercise 5. (*)

For $\Im w > 0$ we set $\Lambda = \mathbb{Z} + w\mathbb{Z}$ and $E_w = \mathbb{C}/\Lambda$.

- (1) Show that if there exists $a \in \mathbb{C} \setminus \mathbb{R}$ with $a\Lambda \subset \Lambda$, then w is imaginary quadratic (algebraic of degree 2 over \mathbb{Q}).
- (2) Conversely, if w is imaginary quadratic, construct a non-real a with $a\Lambda \subset \Lambda$.
- (3) Define $\text{End}(E_w) = \{a \in \mathbb{C} : a\Lambda \subset \Lambda\}$. Deduce that either $\text{End}(E_w) = \mathbb{Z}$ or

$$\text{End}(E_w) = \{n + ma : n, m \in \mathbb{Z}\}$$

for some non-real a as above.